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APPLICATIONS OF THE COVARIANT DERIVATIVE OF CARTAN IN THE CALCULUS OF VARIATIONS

A DISSERTATION SUBMITTED TO THE FACULTY OF THE DIVISION OF THE PHYSICAL SCIENCES IN CANDIDACY FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

1939

By

ELLEN CLAYTON STOKES

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APPLICATIONS OF THE COVARIANT DERIVATIVE OF CARTAN IN THE CALCULUS OF VARIATIONS

Introduction. The purpose of this paper is to express in tensor form the criteria for a minimizing arc for a parametric problem of the calculus of variations with fixed end-points. The problem to be considered is in the n-dimensional space of points $y = (y^1, \ldots, y^n)$ and in the form developed by Wren [8]*. It is that of finding in a class of admissible arcs one which minimizes an integral of the form

$$I = \int_{t_1}^{t_2} [2g(y^1, \ldots, y^n, y^{1'}, \ldots, y^{n'})]^{1/2} dt = \int_{t_1}^{t_2} [2g(y,y')]^{1/2} dt,$$

the function $g(y,y')$ being positively homogeneous of order two in the variables y'.

For a tensor treatment of functions which involve derivatives it is desirable to have a definition of an absolute differential and of related covariant derivatives. Several different covariant derivatives have been suggested in the literature, notably those of Ricci [1] for Riemannian spaces, and of Taylor [6], Berwald [7], and Cartan [9] for Finsler spaces [4]. Of these that of Cartan has been chosen for use in this paper since it includes the others as special cases and seems to be the most general.

In Section 1 a formulation of Wren's problem is given. The condition is imposed that the integrand function $(2g)^{1/2}$ be greater

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*Numbers in square brackets refer to the bibliography at the end of the paper.

than zero in a certain region R of yy'-space, but a proof is presented to show that this is not a real restriction, at least when R is a restricted neighborhood of a given curve C.

In the next two sections the fundamental tensors to be employed are introduced and properties characterizing the absolute differential of Cartan are given. These properties differ somewhat from those which Cartan himself gave. He developed his differential primarily for the study of Finsler spaces. In the present paper the definition is related more directly to problems of the calculus of variations.

In Section 4 Wren's first variation and his Euler differential equations have been transformed to a tensor form involving Cartan's covariant derivative. There, as in Sections 6 and 7 where the second variation and the Jacobi differential equations are discussed, the results agree with those of Householder [10] for the variable end-point problem. Section 5 deals with the invariant properties of the Weierstrass E-function and the Legendre quadratic form. In Section 8 we have found the transformation formula for the determinant whose zeros determine the conjugate points of the minimizing arc. Finally in Section 9 the tensor properties of the derivatives of the Hamiltonian function are presented, and the canonical differential equations are expressed in terms of covariant derivatives.

Throughout the paper the summation convention of tensor analysis has been freely used, namely, that in a product a repeated index indicates a summation. For example, $\ell_i y^i$ stands for $\sum_{i=1}^{n} \ell_i y^i$.

1. The integral function and its properties. We first present a well-known form of the parametric problem of the calculus of variations with fixed end-points. The integrand considered in n-dimensional space is

$$(1:1) \qquad I = \int_{t_1}^{t_2} f(y^1, \ldots, y^n, y^{1'}, \ldots, y^{n'})dt = \int_{t_1}^{t_2} f(y,y')dt$$

and the arcs C along which it is to be evaluated are defined by functions of the form

$$(1:2) \qquad y^i(t) \qquad (t_1 \leqq t \leqq t_2;\ i = 1,\ldots,n).$$

We suppose that there is a region R in the 2n-dimensional space of points (y,y') with the following properties:

(a) $y^{i'}y^{i'} \neq 0$ at all points (y,y') in R,

(b) if (y,y') is in R then (y,ky') is also in R for every $k > 0$,

(c) $f(y,y')$ satisfies the homogeneity relation $f(y,ky') = kf(y,y')$ for every $k > 0$ and (y,y') in R,

(d) $f(y,y')$ has continuous partial derivatives of at least the fourth order in R,

(e) $f(y,y') > 0$ for (y,y') in R,

(f) the matrix $\| \partial^2 f/\partial y^{i'} \partial y^{j'} \|$ is of rank $n-1$ for (y,y') in R.

This last requirement makes the calculus of variations problem to be considered non-singular in R.

All elements (y,y') interior to R will be termed admissible elements. An arc defined by (1:2) will be said to be regular if the functions $y^i(t)$ possess continuous derivatives $y^{i'}$ not vanishing simultaneously on the t-interval belonging to the arc. An admissible arc is one whose elements (y,y') are admissible and which is continuous and made up of a finite number of regular sub-arcs on the interval t_1t_2.

The problem is that of finding in the class of admissible arcs joining two given fixed points 1 and 2 in the n-dimensional space of points $y = (y^1, \ldots, y^n)$ one which minimizes the integral (1:1).

We now transform this problem to the one studied by Wren [8] by expressing the integrand in terms of the function $g(y,y')$ defined by the equation

$$2g(y,y') = f^2(y,y'),$$

so that the integral (1:1) becomes

$$I = \int_{t_1}^{t_2} [2g(y,y')]^{1/2} dt. \tag{1:3}$$

The function $g(y,y')$ will possess in R all the properties of $f(y,y')$ with the exception that (c) and (f) are now, respectively, equivalent to

(c') $g(y,ky') = k^2 g(y,y')$ for every $k > 0$,

(f') $|\partial^2 g/\partial y^{i'} \partial y^{j'}| \neq 0$ in R.

The last statement can be verified easily by means of the definition of $g(y,y')$, the homogeneity property of $f(y,y')$, and well-known matrix theory.

We list here some of the consequences of property (c') which will be of use in the study which follows [8, p. 168]:

$$\begin{aligned} &g_i y^{i'} = 2g, && g_{ijk} y^{k'} = 0, \\ &g_{ij} y^{j'} = g_i, && g_{iy^k} y^{i'} = 2g_{y^k}, \\ &g_{ij} y^{i'} y^{j'} = 2g, && g_{ijy^k} y^{j'} = g_{iy^k}, \end{aligned} \tag{1:4}$$

where

$$g_i = \partial g/\partial y^{i'}, \qquad g_{ij} = \partial^2 g/\partial y^{i'} \partial y^{j'}, \qquad g_{y^k} = \partial g/\partial y^k.$$

These notations will be used throughout the paper.

The condition (e) that $f(y,y') > 0$ in R is not a real restriction upon the nature of the calculus of variations problem described above, at least when the region R is a restricted neighborhood of a given curve C of class C^V[2, p. 13]. For suppose $f(y,y')$ not always positive on the arc C but that R and f otherwise satisfy the conditions (a) - (f). As will be shown below, there exists a function $\varphi(y)$, having continuous fifth derivatives, whose derivative $d\varphi/dt$ will be positive along the given arc C and in a restricted neighborhood of the elements (y,y') on C. The integrand $f(y,y')$ may then be replaced in the integral I of equation (1:1) by $F(y,y') = f(y,y') + k^2 d\varphi/dt$, which for a sufficiently large value of k will be positive along C since $d\varphi/dt$ is positive on C. This function $F(y,y')$ satisfies all the properties (a) - (f) in a neighborhood R of the elements (y,y') on C, as one may readily verify. Moreover, such a change of integrand function does not affect the extremals for I, since the Euler differential equations are identically zero for a derivative function $d\varphi/dt$[2, p. 153].

We now proceed to prove the existence of the function $\varphi(y)$. Suppose that the arc C joining the points 1 and 2 is defined by functions

$$y^i(\xi) \qquad (i = 1,\ldots,n)$$

of class C^V, where ξ is the length of arc measured along C from the point 1. Suppose further that C does not intersect itself between the points 1 and 2 which correspond to $\xi = 0$ and $\xi = \xi_2$, respectively. The derivatives $y^{i\prime}(\xi)$ are the components of the vector tangent to C. Let

$$v_1^i(\xi), \ldots, v_{n-1}^i(\xi) \qquad (i = 1, \ldots, n)$$

be the components of class C^ν of $n-1$ vectors which with the tangent vector form a set of n mutually orthogonal vectors at the point ξ on the curve, so that the determinant

$$(1:5) \qquad \begin{vmatrix} y^{1\prime}(\xi) & v_1^1(\xi) & \cdots & v_{n-1}^1(\xi) \\ \cdots & \cdots & \cdots & \cdots \\ y^{n\prime}(\xi) & v_1^n(\xi) & \cdots & v_{n-1}^n(\xi) \end{vmatrix}$$

is different from zero on the arc C. The equations

$$(1:6) \qquad y^i = y^i(\xi) + \zeta_j v_j^i(\xi) \qquad (i = 1,\ldots,n;\ j = 1,\ldots,n-1)$$

define a transformation of coordinates from $(y^1, \ldots, y^n)$ to $(\xi, \zeta_1, \ldots, \zeta_{n-1})$. It is apparent that to every set (ξ, ζ) there corresponds one set (y).

Let us consider the solution of these equations for ξ and the ζ_i's in terms of the y^i's. We know that there is one set of solutions

$$(y^1, \ldots, y^n, \xi, \zeta_1, \ldots, \zeta_{n-1}) = [y^1(\xi), \ldots, y^n(\xi), \ ,0, \ldots, 0]$$

belonging to the arc C. Moreover, no two distinct solutions of this set can have the same projection in the y-space, since by hypothesis the curve does not intersect itself. In addition, the fact that the functional determinant of equations (1:6) with respect to ξ, $\zeta_1, \ldots, \zeta_{n-1}$ is different from zero on the solutions given above follows from the non-vanishing of the determinant (1:5). Therefore, by implicit function theory [3, pp. 8-11] the equations (1:6) have unique solutions in a restricted neighborhood of the curve C and we may write

$$\xi = \varphi(y^1, \ldots, y^n) = \varphi(y)$$

(1:7)

$$\eta_i = \psi_i(y^1, \ldots, y^n) \qquad (i = 1, \ldots n-1).$$

These functions will be of class at least C^v if the arc C is of class C^v.

Now in the ξ, η space, the curve C becomes the segment $0\xi_2$ on the ξ-axis, and we know that along this segment ξ itself increases monotonically from 0 to ξ_2. Therefore $\varphi(y)$ is a function whose derivative is positive along C and, by continuity, at least in a small neighborhood of the elements (y,y') on C.

2. Fundamental tensors. The type of transformation of coordinates which is to be considered in this paper is defined by equations

(2:1) $$\bar{y}^i = \bar{y}^i(y^1, \ldots, y^n) \qquad (i = 1, \ldots, n)$$

where $\bar{y} = (\bar{y}^1, \ldots, \bar{y}^n)$ is the set of new coordinates. It is supposed that the equations (2:1) are of class C^v in the projection S in y-space of the region R, and that they define with equations

(2:2) $$\bar{y}^{i\prime} = y^{k\prime}\partial\bar{y}^i/\partial y^k$$

a one to one correspondence between the region R and a region $\bar{R}$ of points $(\bar{y},\bar{y}')$. Furthermore we assume that the determinant $|\partial\bar{y}^i/\partial y^k|$ is different from zero in R. According to well-known implicit function theory the inverse of the transformation (2:1), namely,

$$y^i = y^i(\bar{y}^1, \ldots, \bar{y}^n) \qquad (i = 1, \ldots, n)$$

is then also of class C^{ν} in $\bar{R}$.

Sets of values Y^i, Y_i associated with the point y which are transformed into sets $\bar{Y}^i$, $\bar{Y}_i$ at the point $\bar{y}$ by the transformations

$$\bar{Y}^i = Y^k \partial \bar{y}^i / \partial y^k, \qquad \bar{Y}_i = Y_k \partial y^k / \partial \bar{y}^i \tag{2:3}$$

are called, respectively, contravariant and covariant tensors of rank one. It follows readily by differentiating (2:1) that the derivatives $y^{i'}$ associated with a curve $y^i(t)$ satisfy (2:2) and are the components of a contravariant tensor.

We assume that the function $g(y,y')$ is an invariant so that by definition

$$\bar{g}(\bar{y},\bar{y}') = g(y,y').$$

It then follows by differentiating this equation that g_i, g_{ij}, and g_{ijk} are the components of covariant tensors of ranks one, two, and three, respectively. We define g^{ij} to be the element corresponding to g_{ij} in the reciprocal of the determinant $|g_{ij}|$, so that $g^{ij}g_{ik} = \delta_k^{\ j}$ where $\delta_k^{\ j}$ is 1 if $j = k$ and 0 if $j \neq k$. Then g^{ij} is also a tensor of rank 2 [5, p. 43]. The tensors g_{ij} and g^{ij} are often used for lowering and raising indices. The tensors ℓ^i and $\ell_i = g_{ij}\ell^j$ are called corresponding contravariant and covariant tensors.

We call attention to the fact that g_{y^k} is not a tensor, as is seen from the formula

$$\begin{aligned}\bar{g}_{\bar{y}^k} &= g_{y^r} \partial y^r / \partial \bar{y}^k + g_s \partial y^{s'} / \partial \bar{y}^k \\ &= g_{y^r} \partial y^r / \partial \bar{y}^k + g_s \left(\partial^2 y^s / \partial \bar{y}^k \partial \bar{y}^h \right) y^{r'} \partial \bar{y}^h / \partial y^r .\end{aligned}$$

As usual we define the length of a vector of components Y^i to be $(g_{ij}Y^iY^j)^{1/2}$, and the cosine of the angle Θ between the vectors of components Y^i and X^j to be

$$\cos\Theta = g_{ij}X^iY^j/(g_{hk}X^hX^k)^{1/2}(g_{rs}Y^rY^s)^{1/2}.$$

From the definition of vector length just given it follows that the numbers

$$\ell^i = y^{i'}/(g_{kj}y^{k'}y^{j'})^{1/2} = y^{i'}/(2g)^{1/2} \tag{2:4}$$

are the components of a contravariant unit vector in the direction defined by the $y^{i'}$, and that the numbers

$$\ell_i = g_{ij}y^{j'}/(g_{hk}y^{h'}y^{k'})^{1/2} = g_i/(2g)^{1/2} \tag{2:5}$$

are the components of the corresponding covariant vector. Moreover

$$\ell^i\ell_i = g_{ij}y^{i'}y^{j'}/2g = 1.$$

3. The absolute differential of Cartan. The following postulates, somewhat different from those of Cartan [9, p. 10], are sufficient to define his absolute differential for which the symbol D will be used below.

(A) The absolute differential of an invariant is the ordinary differential.

(B) DY^i is a function of the variables y, y', dy, dy', Y, dY, linear in Y and dY so that

$$D(cY^i) = cDY^i, \qquad D(Y^i + Z^i) = DY^i + DZ^i.$$

From this property it follows that the elements DY^i have the form

$$DY^i = M^i_k\,dY^k + C^i_kY^k.$$

(C) DX_i has similar properties relative to X_i and dX_i, so that

$$DX_i = L_i^k\, dX_k + B_i^k X_k.$$

(D) For all outer and inner products of tensors the usual distributive law of the calculus holds, for example

$$D(X_i Y^i) = X_i DY^i + Y^i DX_i.$$

If Y^i is the tensor whose components in y-space are δ^i_j for a fixed j, it follows from the fact that $D(X_i Y^i) = d(X_i Y^i)$ that the matrices L_i^k and M_k^i are the identity matrices and the equations

$$DX_i = dX_i - C_i^k X_k,$$

$$DY^i = dY^i + C_k^i Y^k$$

are valid.

(E) The elements of DY^i are polynomials in dy^i and $dy^{i\prime}$ and are linear and homogeneous of order one in these arguments. It then follows that we may write

$$\begin{aligned} DY^i &= dY^i + Y^k(\Gamma_{k\ h}^{\ i}\, dy^h + C_{k\ h}^{\ i}\, dy^{h\prime}), \\ DX_i &= dX_i + X_k(\Gamma_{i\ h}^{\ k}\, dy^h + C_{i\ h}^{\ k}\, dy^{h\prime}). \end{aligned} \tag{3:1}$$

(F) $g_{ij}Y^iZ^j$ is constant when $DY^i = DZ^i = 0$.

(G) $C_{ijh} = C_{jih}$.

From the property (F) it follows that

$$d(g_{ij}Y^iZ^j) = Y^iZ^j dg_{ij} + g_{ij}Z^j dY^i + g_{ij}Y^i dZ^j = 0$$

when $DY^i = DZ^j = 0$, so that from (3:1)

$$dg_{ij} = (\Gamma_{ijh} + \Gamma_{jih})dy^h + (C_{ijh} + C_{jih})dy^{h'},$$

and hence

$$\begin{aligned} g_{ij\,y^h} &= \Gamma_{ijh} + \Gamma_{jih}, \\ g_{ijh} &= C_{ijh} + C_{jih}. \end{aligned} \tag{3:2}$$

Then from property (G) we have

$$C_{ijh} = \tfrac{1}{2} g_{ijh} \tag{3:3}$$

and therefore C_{ijh} is symmetric in all three indices.

It follows now from equations (1:4) that $C_{ijh}y^{i'} = 0$ and hence from (3:1) that $Dy^{i'}$ takes the form

$$Dy^{i'} = dy^{i'} + y^{k'}\Gamma^{\ i}_{k\ h}dy^h. \tag{3:4}$$

With the aid of this equation the equations (3:1) may be written in the form

$$\begin{aligned} DY^i &= dY^i + Y^k(\Gamma^{*\ i}_{k\ h}dy^h + C_k^{\ i}{}_h Dy^{h'}), \\ DX_i &= dX_i - X_k(\Gamma^{*\ k}_{i\ h}dy^h + C_i^{\ k}{}_h Dy^{h'}), \end{aligned} \tag{3:5}$$

where

$$\Gamma^{*\ k}_{i\ h} = \Gamma^{\ k}_{i\ h} - C_i^{\ k}{}_r y^{m'}\Gamma^{\ r}_{m\ n}. \tag{3:6}$$

The final property assumed for the absolute differential is the further symmetry property

(H) $\Gamma^{*\ i}_{k\ h} = \Gamma^{*\ i}_{h\ k}$.

From these postulates we may find, with Cartan [9, pp. 14-16], the following values of the coefficients Γ and Γ^*:

$$\Gamma_{kjh} = \gamma_{kjh} + C_{khr}\,\partial G^r/\partial y^{j'} - C_{jhr}\,\partial G^r/\partial y^{k'},$$

(3:7)

$$\Gamma^*_{kjh} = \gamma_{kjh} + C_{khr}\,\partial G^r/\partial y^{j'} - C_{kjr}\,\partial G^r/\partial y^{h'} - C_{jhr}\,\partial G^r/\partial y^{k'},$$

where

$$\gamma_{kjh} = (g_{kjy^h} + g_{jhy^k} - g_{khy^j})/2,$$

$$G_m = (g_{my^k}y^{k'} - g_{y^m})/2.$$

As we shall see later, the Euler differential equations of Wren take the form

$$dg_i/dt - g_{y^i} = g_{ij}y^{j''} + g_{iy^j}y^{j'} - g_{y^i} = 0.$$

By means of the value of G_m given above these equations become

$$g_{ij}y^{j''} + 2G_i = 0,$$

which, upon contraction with g^{ik}, are seen to be equivalent to

$$y^{k''} + 2G^k = 0.$$

We also make note of Cartan's equation X [9, p. 16] which will be of use to us later

(3:8)
$$y^{h'}\Gamma^{\ i}_{h\ j} = \partial G^i/\partial y^{j'}.$$

The first member of this equation is equal to $y^{h'}\Gamma^{*i}_{h\ j}$, as may readily be verified by means of the relations (1:4).

A most important result of the above definition is the fact that the absolute differential of a covariant or of a contravariant tensor is itself a tensor of the same kind. This can be shown by a transformation of coordinates of the kind described in Section 2. The proof is too long to be reproduced here in detail,

so that we shall give only the main steps in the argument.

Suppose that a bar over a function indicates that the arguments are $\bar{y}$ and $\bar{y}'$. We wish to prove that $\overline{DY^i} = DY^\alpha \partial \bar{y}^i / \partial y^\alpha$. Let us consider the laws of transformation of the elements of

$$\overline{DY^i} = d\bar{Y}^i + \bar{Y}^k(\overline{\Gamma}_{k\,h}^{\;i}\, dy^h + \bar{C}_{k\,h}^{\;i}\, d\bar{y}^{h'}). \tag{3:9}$$

Differentiating the expression for Y^i in equation (2:3) we find that

$$d\bar{Y}^i = dY^\alpha \partial \bar{y}^i / \partial y^\alpha + Y^\alpha\, dy^\beta\, \partial^2 \bar{y}^i / \partial y^\alpha \partial y^\beta . \tag{3:10}$$

Since $\bar{y}^{h'}$ is a contravariant tensor, $d\bar{y}^{h'}$ transforms by the same formula (3:10). The coefficient $\bar{C}_{k\,h}^{\;i}$ is a mixed tensor since it is the inner product of the tensor g^{ij} by the tensor C_{kjh}, and dy^h is also a tensor. Furthermore we find after a somewhat extended computation that the elements of $\overline{\Gamma}_{k\,h}^{\;i}$ transform as follows:

$$\begin{aligned}
\bar{G}_r &= (1/2)[g_{\beta\,y^\alpha}\, y^{\alpha'} \partial y^\beta / \partial \bar{y}^r \\
&\quad + g_{\beta\alpha}\, y^{\sigma'}(\partial y^\beta / \partial \bar{y}^r)(\partial \bar{y}^s / \partial y^\sigma)(\partial y^{\alpha'} / \partial \bar{y}^s) - g_{y^\alpha}\, \partial y^{\alpha'} / \partial \bar{y}^r], \\
\bar{C}_{khr}\, \partial \bar{G}^r / \partial \bar{y}^{j'} &= C_{\kappa\lambda\rho}(\partial G^\rho / \partial y^{\nu'})(\partial y^\kappa / \partial \bar{y}^k)(\partial y^\eta / \partial \bar{y}^h)(\partial y^\nu / \partial \bar{y}^j) \\
&\quad + C_{\kappa\alpha\eta}(\partial y^\kappa / \partial \bar{y}^k)(\partial y^\eta / \partial \bar{y}^h)(\partial y^{\alpha'} / \partial \bar{y}^j), \\
\bar{\gamma}_{kjh} &= \gamma_{\kappa\nu\eta}(\partial y^\kappa / \partial \bar{y}^k)(\partial y^\nu / \partial \bar{y}^j)(\partial y^\eta / \partial \bar{y}^h) \\
&\quad + g_{\kappa\nu}(\partial y^\nu / \partial \bar{y}^j)(\partial^2 y^\kappa / \partial \bar{y}^k \partial \bar{y}^h) \\
&\quad + C_{\kappa\nu\eta}[(\partial y^\kappa / \partial \bar{y}^k)(\partial y^\nu / \partial \bar{y}^j)(\partial y^{\eta'} / \partial \bar{y}^h) \\
&\quad + (\partial y^{\kappa'} / \partial \bar{y}^k)(\partial y^\nu / \partial \bar{y}^j)(\partial y^\eta / \partial \bar{y}^h) \\
&\quad - (\partial y^\kappa / \partial \bar{y}^k)(\partial y^{\nu'} / \partial \bar{y}^j)(\partial y^\eta / \partial \bar{y}^h)].
\end{aligned} \tag{3:11}$$

Then an application of these equations (3:11) and the statements preceding them to the first of equations (3:7) shows that

$$\Gamma^{i}_{k\,h} = \Gamma_{\kappa}{}^{\mu}{}_{\eta}(\partial y^{\kappa}/\partial \bar{y}^{k})(\partial y^{\eta}/\partial \bar{y}^{h})(\partial \bar{y}^{i}/\partial y^{\mu}) + (\partial^{2} y^{\mu}/\partial \bar{y}^{k}\partial \bar{y}^{h})(\partial \bar{y}^{i}/\partial y^{\mu})$$

(3:12)

$$+ C_{\kappa}{}^{\mu}{}_{\eta}(\partial y^{\kappa}/\partial \bar{y}^{k})(\partial y^{\eta'}/\partial \bar{y}^{h})(\partial \bar{y}^{i}/\partial y^{\mu}).$$

Finally, with the aid of the above laws of transformation, we see that

$$\overline{DY^{i}} = (\partial \bar{y}^{i}/\partial y^{\mu})[dY^{\mu} + Y^{\kappa}(\Gamma_{\kappa}{}^{\mu}{}_{\eta}\,dy^{\eta} + C_{\kappa}{}^{\mu}{}_{\eta}\,dy^{\eta'})],$$

which proves the tensor property. By these same formulas it also follows that DX_1 is a covariant tensor.

The definition of the absolute differential can be extended so that it will apply to tensors of higher order than the first. In order to make this extension let us first consider the special covariant tensor $T_{ij} = X_i Y_j$. Then by postulate (D) and equations (3:1) we have

$$\begin{aligned} DT_{ij} &= X_i DY_j + Y_j DX_i, \\ &= X_i dY_j + Y_j dX_i - X_i Y_k \Gamma_{j\,h}^{k} dy^{h} - Y_j X_k \Gamma_{i\,h}^{k} dy^{h} \\ &\quad - X_i Y_k C_{j\,h}^{k} dy^{h'} - Y_j X_k C_{i\,h}^{k} dy^{h'} \\ &= dT_{ij} - T_{ik}\Gamma_{j\,h}^{k} dy^{h} - T_{kj}\Gamma_{i\,h}^{k} dy^{h} - T_{ik} C_{j\,h}^{k} dy^{h'} \\ &\quad - T_{kj} C_{i\,h}^{k} dy^{h'}. \end{aligned} \tag{3:13}$$

This will be taken as the definition of DT_{ij} for an arbitrary tensor T_{ij}. It can readily be shown to be itself a covariant tensor of the second rank. In the same way we find that if $T_i{}^{j} = X_i Y^{j}$ then

$$DT_i^{\ j} = dT_i^{\ j} - T_k^{\ j}\Gamma_{i\ h}^{\ k}dy^h + T_i^{\ k}\Gamma_{k\ h}^{\ j}\,dy^h$$
$$- T_k^{\ j}C_{i\ h}^{\ k}dy^h - T_i^{\ k}C_{k\ h}^{\ j}\,dy^{h'},$$

and for a tensor T^{ij}

$$DT^{ij} = dT^{ij} + T^{ik}\Gamma_{k\ h}^{\ j}\,dy^h + T^{kj}\Gamma_{k\ h}^{\ i}\,dy^{h'}$$
$$+ T^{ik}C_{k\ h}^{\ j}\,dy^{h'} + T^{kj}C_{k\ h}^{\ i}\,dy^{h'}. \qquad (3:14)$$

Proceeding in this way we may find the form for the absolute differential of any tensor $T^{s_1\cdots s_e}_{r_1\cdots r_m}$ and the result will be a tensor of the same kind.

An immediate result of equations (3:2) and (3:13) is that

$$Dg_{ij} = 0 \qquad (3:15)$$

and similarly, with the aid of the equation $g^{ij}g_{jk} = \delta^i_k$,

$$Dg^{ij} = 0. \qquad (3:16)$$

4. The first variation and the Euler differential equations. In constructing the first and second variations Wren [8, p. 168] considers the one-parameter family of curves

$$y^i(t,a) = y^i(t) + a\eta^i(t) \qquad (t_1 \leqq t \leqq t_2;\ i = 1, \ldots, n), \qquad (4:1)$$

where the $\eta^i(t)$ are a set of admissible variations satisfying the conditions

$$\eta^i(t) \text{ of class } D' \text{ for } t_1 \leqq t \leqq t_2,$$
$$\eta^i(t_1) = \eta^i(t_2) = 0.$$

The family contains the curve C defined by functions (1:2) when $a = 0$. The η^i are the components of a contravariant tensor since

$$\eta^i = \partial [y^i(t,a)]/\partial a, \qquad \partial \bar{y}^i/\partial a = (\partial \bar{y}^i/\partial y^\alpha)(\partial y^\alpha/\partial a)$$

and hence

$$\eta^i(t) = \eta^\alpha \partial \bar{y}^i/\partial y^\alpha.$$

The first variation as found by Wren is

$$(4:2) \qquad I'(0) = \int_{t_1}^{t_2} (g_{yi}\eta^i + g_1\eta^{i'})(2g)^{-1/2}dt.$$

Since the derivative of an invariant is an invariant, it is clear that the integrand function, which is the partial derivative of $(2g)^{1/2}$ with respect to a, is an invariant. However, we may express this integrand in terms of tensors in such a way that from its very form its invariance is at once apparent. This we proceed to do.

With the aid of the homogeneity relations (1:4) the expression for Γ^*_{hmi} in equation (3:7) provides the relationship

$$(4:3) \qquad y^{h'}y^{m'}\Gamma^*_{hmi} = g_{yi}.$$

Now if in the integrand function we substitute this expression for g_{yi} together with the expression for $\eta^{i'}$ found in the equation

$$D\eta^i/dt = \eta^{i'} + \eta^k \Gamma^*{}^i_{k\,h} y^{h'} + \eta^k C_k{}^i{}_h Dy^{h'}/dt$$

we find that

$$g_{yi}\eta^i + g_1\eta^{i'} = y^{m'}y^{h'}\Gamma^*_{hmi}\eta^i + g_1 D\eta^i/dt - g_1\eta^k\Gamma^*{}^i_{h\,k}y^{h'} - g_1\eta^k C_k{}^i{}_h Dy^{h'}/dt.$$

By means of equations (1:4), (3:3) and property (H) we derive the relationships

$$g_i \eta^k \Gamma^{*\,i}_{k\,h} y^{h'} = y^{m'} y^{h'} \Gamma^{*}_{hmk} \eta^k,$$

(4:4)

$$g_i C_k^{\ i}{}_h = y^{m'} C_{kmh} = 0.$$

Substituting from the last three equations in (4:2) and using relation (2:5) we deduce easily

THEOREM 4:1. The integral

$$I(a) = \int_{t_1}^{t_2} [2g(y + a\eta,\ y' + a\eta')]^{1/2} dt$$

has for its first derivative at $a = 0$ the form

$$I'(0) = \int_{t_1}^{t_2} \ell_i (D\eta^i/dt) dt \tag{4:5}$$

where the ℓ_i are the covariant components of the unit vector tangent to the curve C.

The Euler equations of Wren are [8, p. 169]

$$P_i \equiv d[g_i (2g)^{-1/2}]/dt - g_{yi}(2g)^{-1/2} = 0 \qquad (i = 1, \ldots, n), \tag{4:6}$$

or when the parameter is so specialized that $(2g)^{1/2} = 1$

$$dg_i/dt - g_{yi} = 0 \qquad (i = 1, \ldots, n).$$

Taking the absolute derivative of $g_i (2g)^{-1/2} = \ell_i$ with respect to t we have, from equations (3:5) and (4:4),

$$D\ell_i/dt = d\ell_i/dt - \ell_k \Gamma^{*\,k}_{i\,h} y^{h'}.$$

This together with equation (4:3) gives

THEOREM 4:2. The Euler differential equations

$$P_i \equiv d[g_i (2g)^{-1/2}]/dt - g_{yi}(2g)^{-1/2} = 0 \qquad (i = 1, \ldots, n)$$

for the integral I, equation (1:3), are identical with the equations

$$(4:7) \qquad D\ell_i/dt = 0 \qquad (i = 1, \ldots, n)$$

and hence the P_i are the components of a covariant tensor.

There is a second form for the P_i displayed by Wren [8, p. 171], namely,

$$(4:8) \qquad P_i = (2gg_{ij} - g_i g_j)Q^j(2g)^{-3/2}$$

where

$$Q^i = y^{i''} + \{jk,i\}\, y^{j'}y^{k'}.$$

This form, though different in appearance, is exactly the same as $D\ell_i/dt$, as we shall now show. Wren's symbol $\{jk,i\}$ has the same value as our $\gamma_{jk}^{\ i}$, and when $\Gamma^*_{jk}{}^i y^{j'}y^{k'}$ is found from equation (3:7) it is seen to be just $\gamma_{jk}^{\ i} y^{j'}y^{k'}$. Hence equation(3:4) shows that

$$(4:9) \qquad Q^j = Dy^{j'}/dt.$$

By means of equation (2:5) and postulate (A) it is readily verified that

$$(4:10) \qquad D\ell_i/dt = (2g)^{-1/2}Dg_i/dt - g_i(2g)^{-3/2}dg/dt.$$

But from equations (1:4) $2g = g_j y^{j'}$ and hence

$$2dg/dt = g_j Dy^{j'}/dt + y^{j'}Dg_i/dt.$$

Moreover, since by (1:4) $g_i = g_{ij}y^{j'}$, it follows with the aid of equation (3:15) that

$$Dg_i/dt = g_{ij}Dy^{j'}/dt.$$

Substituting these last two results in equation (4:10) we see that

$$D\,\ell_i/dt = (2gg_{ij} - g_i g_j)(2g)^{-3/2} Dy^{j'}/dt$$

which by equation (4:9) is the same as (4:8).

We note here for future use the fact that equations (4:7) are equivalent to

$$D\,\ell^i/dt = 0 \tag{4:11}$$

since $\ell_i = g_{ij}\,\ell^j$ and $Dg_{ij}/dt = 0$.

5. The Weierstrass E-function and Legendre's quadratic form. The E-function and the quadratic form as given by Wren [8, p. 174] are

$$E(y,y',Y') = [2g(y,y')]^{1/2} - [g_i(y,y')Y^{i'}][2g(y,y')]^{-1/2}, \tag{5:1}$$

$$\begin{aligned} Q(y,y',\xi) &= g_{ij}(y,y')\,\xi^i\,\xi^j[2g(y,y')]^{-1/2} \\ &\quad - [g_i(y,y')\xi^i]^2[2g(y,y')]^{-3/2}. \end{aligned} \tag{5:2}$$

The Weierstrass E-function is evidently an invariant if the $Y^{i'}$ are the components of a contravariant tensor and they may be so taken since they define a direction. The ξ^i in $Q(y,y',\xi)$ are variations like the η^i and hence may also be considered as the components of a contravariant tensor. We state the result as

THEOREM 5:1. *The functions* $E(y,y',Y')$ *and* $Q(y,y',\xi)$ *defined by equations* (5:1) *and* (5:2) *are invariants when the functions* $Y^{i'}$ *and* ξ^i *are the components of contravariant tensors of rank one.*

6. The second variation. According to Wren [8, p. 169] the second variation has the form

$$I''(0) = \int_{t_1}^{t_2} [2\omega(2g)^{-1/2} - (g_{yi}\eta^i + g_i\eta^{i'})^2(2g)^{-3/2}]\,dt$$

where

$$2\omega = g_{y^i y^j}\eta^i\eta^j + 2g_{iy^j}\eta^{i'}\eta^j + g_{ij}\eta^{i'}\eta^{j'}. \tag{6:1}$$

Along an extremal where equations (4:6) are satisfied $I''(0)$ becomes

$$I''(0) = \int_{t_1}^{t_2}[2\omega - (\ell_1\eta^i)'^2](2g)^{-1/2}dt. \tag{6:2}$$

We now propose to reduce the expression for 2ω to one involving tensors so that, as is the case with $(\ell_1\eta^i)'$, its invariance will be at once obvious from its form. The first step in the somewhat long reduction will be that of replacing the $\eta^{i'}$ where they occur in 2ω by expressions obtained for them by means of the equations

$$D\eta^i/dt = \eta^{i'} + \eta^k\Gamma^*_{k\ h}{}^i{}_{y^{h'}} + \eta^k C_{k\ h}{}^i{}_{Dy^{h'}}/dt.$$

Now along an extremal arc E_{12} where $D\ell^h/dt = 0$, the last term drops out since then

$$D\ell^h/dt = D[y^{h'}(2g)^{-1/2}]/dt = (Dy^{h'}/dt)(2g)^{-1/2} + y^{h'}d(2g)^{-1/2}/dt = 0$$

in which we have used the fact that the absolute derivative of an invariant is equal to the ordinary derivative. Hence by (3:3) and the homogeneity property (1:4) it follows from the last equation that along an extremal $C_k{}^i{}_h Dy^{h'}/dt = 0$ and we may write

$$D\eta^i/dt = \eta^{i'} + \eta^k\Gamma^*_{k\ h}{}^i{}_{y^{h'}}. \tag{6:3}$$

Substituting the value of $\eta^{i'}$ from this equation in (6:1) we obtain

$$\begin{aligned} 2\omega = {} & \eta^i\eta^j[g_{y^iy^j} - 2g_{ky^j}\Gamma^*_{i\ h}{}^k{}_{y^{h'}} + g_{kr}\Gamma^*_{i\ h}{}^r\Gamma^*_{j\ m}{}^k{}_{y^{h'}y^{m'}}] \\ & + 2(D\eta^i/dt)\eta^j(g_{iy^j} - g_{ir}\Gamma^*_{j\ h}{}^r{}_{y^{h'}}) \\ & + g_{ij}(D\eta^i/dt)(D\eta^j/dt), \end{aligned} \tag{6:4}$$

where the last term is an invariant and will be retained in its present form in the final result.

Our next step is to replace the term in $(D\eta^i/dt)\eta^j$ by a derivative of a term with respect to t plus terms in $\eta^i\eta^j$. An expression for g_{iyj} in terms of the functions Γ^* can be deduced from (3:2) multiplied by $y^{i'}$ and summed, (3:5), (3:3), and (1:4). The value so obtained is

$$g_{iyj} = y^{m'}(\Gamma^*_{mij} + \Gamma^*_{imj}). \tag{6:5}$$

It follows from this, because by postulate (H) $\Gamma^*_{mij} = \Gamma^*_{jim}$, and from (6:3) that the second row in the expression (6:4) divided by $(2g)^{1/2}$ has the value

$$\begin{aligned}
&2(D\eta^i/dt)\eta^j y^{m'}\Gamma^*_{imj}(2g)^{-1/2} \\
&= 2\eta^{i'}\eta^j y^{m'}\Gamma^*_{imj}(2g)^{-1/2} + 2\eta^k\eta^j y^{m'}\Gamma^*_{imj}\Gamma^*_{k\ h}{}^i y^{h'}(2g)^{-1/2} \\
&= d[\eta^i\eta^j y^{m'}\Gamma^*_{imj}(2g)^{-1/2}]/dt - \eta^i\eta^j d[y^{m'}\Gamma^*_{imj}(2g)^{-1/2}]/dt \\
&\qquad + 2\eta^k\eta^j y^{m'}\Gamma^*_{imj}\Gamma^*_{k\ h}{}^i y^{h'}(2g)^{-1/2}.
\end{aligned}$$

If we insert this expression for the second row in (6:4) and modify the notation for the quadratic form in $\eta^i\eta^j$ so that its coefficients are symmetric, then (6:4) divided by $(2g)^{1/2}$ takes the form indicated in the following theorem:

THEOREM 6:1. *The quadratic form* $2\omega(2g)^{-1/2}$ *in the integrand function of the second variation may be written in the form*

$$\begin{aligned}
2\omega(2g)^{-1/2} &= [g_{ij}(D\eta^i/dt)(D\eta^j/dt) - \eta^i\eta^j R_{ij}](2g)^{-1/2} \\
&\quad + d[\eta^i\eta^j y^{m'}\Gamma^*_{imj}(2g)^{-1/2}]/dt
\end{aligned} \tag{6:6}$$

where

$$R_{ij} = g_{yiyj} - g_{kyj}\Gamma^{*}{}_{i\ h}^{\ k}y^{h'} - g_{kyi}\Gamma^{*}{}_{j\ h}^{\ k}y^{h'}$$

$$(6:7) \quad + y^{m'}\Gamma^{*}_{jkm}\Gamma^{*}{}_{i\ h}^{\ k}y^{h'} + y^{m'}\Gamma^{*}_{ikm}\Gamma^{*}{}_{j\ h}^{\ k}y^{h'}$$

$$+ y^{m'}\Gamma^{*}_{jmk}\Gamma^{*}{}_{i\ h}^{\ k}y^{h'} - (2g)^{1/2}d[y^{m'}\Gamma^{*}_{imj}(2g)^{-1/2}]/dt$$

is a tensor.

The proof has already been made except for the statement that R_{ij} is a tensor. In order to prove this we show first that the last term in (6:6) is an invariant. Since $y^i(t,a) = y^i(t) + a\eta^i(t)$ we see that

$$g_k(D\eta^k/da) = g_{km}y^{m'}[\partial\eta^k/\partial a + \eta^i\Gamma^{*}{}_{i\ j}^{\ k}(\partial y^j/\partial a) + \eta^i C_{i\ j}^{\ k}(\partial y^{j'}/\partial a)]$$

$$= \eta^i y^{m'}\Gamma^{*}_{imj}\eta^j,$$

the last value being due to the fact that $\partial\eta^k/\partial a = 0$ and $y^{m'}C_{imj} = 0$. Hence the last term in (6:6) is an invariant, and as a result $\eta^i\eta^j R_{ij}$ is also. To prove that R_{ij} is a tensor we apply a transformation (2:1) and find

$$\bar{\eta}^i\bar{\eta}^j\bar{R}_{ij} = \eta^\alpha\eta^\beta R_{\alpha\beta}$$

$$= \bar{\eta}^i\bar{\eta}^j(\partial y^\alpha/\partial\bar{y}^i)(\partial y^\beta/\partial\bar{y}^j)R_{\alpha\beta}.$$

This is an identity for arbitrary elements $\bar{\eta}^i$ and hence, because of the symmetry of R_{ij},

$$\bar{R}_{ij} = R_{\alpha\beta}(\partial y^\alpha/\partial\bar{y}^i)(\partial y^\beta/\partial\bar{y}^j)$$

which shows that R_{ij} is a covariant tensor of rank two. This concludes the proof of the theorem.

As a corollary to Theorem (6:1) we have

THEOREM 6:2. The second variation

$$I''(0) = \int_{t_1}^{t_2}[2\omega - (\ell_i\eta^i)'^2](2g)^{-1/2}dt$$

may be written in the form

$$I''(0) = \int_{t_1}^{t_2} [g_{ij}(D\eta^i/dt)(D\eta^j/dt) - \eta^i\eta^j R_{ij} - (\ell_i\eta^i)'^2](2g)^{-1/2}dt. \tag{6:8}$$

This follows at once from (6:2) and (6:6) because

$$\int_{t_1}^{t_2} [d(\eta^i\eta^j y^{m'}\Gamma^*_{imj}(2g)^{-1/2})/dt]dt = \eta^i\eta^j y^{m'}\Gamma^*_{imj}(2g)^{-1/2}\Big|_{t_1}^{t_2} = 0$$

since $\eta^i(t_1) = \eta^i(t_2) = 0$.

It is interesting to note that the tensor R_{ij} is closely related to the tensor R_{himj} of Cartan [9, p. 36, XIX]. In order to show this we shall express the elements of R_{ij} in (6:7) in terms of the functions Γ^* and their derivatives so that the relationship will be apparent. Let us first consider the derivative

$$\begin{aligned} &d[y^{m'}\Gamma^*_{imj}(2g)^{-1/2}]/dt \\ &= \Gamma^*_{imj}d[y^{m'}(2g)^{-1/2}]/dt + y^{m'}(2g)^{-1/2}[(\partial\Gamma^*_{imj}/\partial y^r)y^{r'} \\ &\qquad + (\partial\Gamma^*_{imj}/\partial y^{r'})y^{r''}]. \end{aligned} \tag{6:9}$$

Now along the extremal arc E_{12} we have $D\ell^m/dt = 0$. Therefore from (2:4), (3:5), and (1:4) it follows that

$$d[y^{m'}(2g)^{-1/2}]/dt = -y^{k'}\Gamma^*_{k\ h}{}^m y^{h'}(2g)^{-1/2},$$

and after expanding the left hand member

$$y^{m''}(2g)^{-1/2} = -y^{k'}\Gamma^*_{k\ h}{}^m y^{h'}(2g)^{-1/2} - y^{m'}d(2g)^{-1/2}/dt.$$

When we substitute from these last two equations the equation (6:9) becomes

$$d[y^{m'}\Gamma^{*}_{imj}(2g)^{-1/2}]/dt$$

$$(6:10) \qquad = (2g)^{-1/2}[-y^{m'}y^{h'}\Gamma^{*}_{m\ h}{}^{k}\Gamma^{*}_{ikh} + y^{m'}y^{h'}(\partial\Gamma^{*}_{ihj}/\partial y^{m}) - y^{m'}y^{h'}(\partial\Gamma^{*}_{ihj}/\partial y^{r'})(\partial G^{r}/\partial y^{m'})]$$

if we make use of (3:8) and the homogeneity of order zero of Γ^{*}_{imj} which follows from (3:7).

Employing (6:10), (6:5), and the result of differentiating (4:3) with respect to y^{j} we are now able to rewrite the expression for R_{ij} in (6:7) as

$$(6:11) \qquad y^{m'}y^{h'}[\partial\Gamma^{*}_{hmi}/\partial y^{j} - \partial\Gamma^{*}_{ihj}/\partial y^{m} + (\partial\Gamma^{*}_{ihj}/\partial y^{r'})(\partial G^{r}/\partial y^{m'}) - \Gamma^{*}_{mkj}\Gamma^{*}_{i\ h}{}^{k} + \Gamma^{*}_{ikj}\Gamma^{*}_{m\ h}{}^{k}].$$

Again applying (6:5) we see that

$$\partial\,[y^{m'}(\Gamma^{*}_{imh} + \Gamma^{*}_{mih})]/\partial y^{j} = g_{iy^{j}y^{h}}$$

and therefore the first term in (6:11) has the value

$$(6:12) \qquad y^{m'}y^{h'}(\partial\Gamma^{*}_{hmi}/\partial y^{j}) = y^{m'}y^{h'}(\partial\Gamma^{*}_{imh}/\partial y^{j}) = -y^{m'}y^{h'}(\partial\Gamma^{*}_{mih}/\partial y^{j}) + g_{iy^{j}y^{h}y^{h'}}.$$

Similarly the second term in (6:11) is

$$(6:13) \qquad y^{m'}y^{h'}(\partial\Gamma^{*}_{ihj}/\partial y^{m}) = -\,y^{m'}y^{h'}(\partial\Gamma^{*}_{hij}/\partial y^{m}) + g_{iy^{j}y^{m}y^{m'}}.$$

To modify the third term in (6:11) we differentiate (6:5) with respect to $y^{r'}$ and apply (3:8), then (3:6) and (3:2) so that we have

$$(6:14) \qquad y^{h'}(\partial\Gamma^{*}_{ihj}/\partial y^{r'}) = 2C_{ikr}(\partial G^{k}/\partial y^{j'}) - y^{h'}(\partial\Gamma^{*}_{hij}/\partial y^{r'}).$$

Furthermore it can be verified by a differentiation with respect to $y^{r'}$ of the Γ^*_{mih} as found in equation (3:7) that

$$\begin{aligned}\partial\Gamma^*_{him}/\partial y^{r'} &= (1/2)(g_{hiry^m} + g_{imry^h} - g_{hmry^i}) + C_{hmrk}(\partial G^k/\partial y^{i'}) \\ &\quad - C_{hirk}(\partial G^k/\partial y^{m'}) - C_{imrk}(\partial G^k/\partial y^{h'}) \\ &\quad + C_{hmk}(\partial^2 G^k/\partial y^{i'}\partial y^{r'}) - C_{hik}(\partial^2 G^k/\partial y^{m'}\partial y^{r'}) \\ &\quad - C_{imk}(\partial^2 G^k/\partial y^{h'}\partial y^{r'}).\end{aligned}$$

Now when we multiply by $y^{m'}y^{h'}$ and add and use the homogeneity of order minus one of C_{hik}, this last equation gives us

$$y^{m'}y^{h'}(\partial\Gamma^*_{mih}/\partial y^{r'}) = 2C_{irk}(\partial G^k/\partial y^{m'})y^{m'}. \tag{6:15}$$

Finally by means of equations (6:12) to (6:15) we obtain in place of (6:11)

$$\begin{aligned}R_{ij} = -y^{h'}y^{m'}\,[&\partial\Gamma^*_{him}/\partial y^j - (\partial\Gamma^*_{mih}/\partial y^{r'})(\partial G^r/\partial y^{j'}) \\ &-\partial\Gamma^*_{hij}/\partial y^m + (\partial\Gamma^*_{hij}/\partial y^{r'})(\partial G^r/\partial y^{m'}) \\ &+\Gamma^*_{hkj}\Gamma^{*\,k}_{i\,m} - \Gamma^*_{ikj}\Gamma^{*\,k}_{m\,h}].\end{aligned}$$

A comparison of this expression with Cartan's equation XIX [9, p. 36] and a use of equations (3:3) and (1:4) give us the following theorem:

THEOREM 6:3. The tensor R_{ij} of equation (6:7) when evaluated along an extremal arc E_{12} is exactly

$$y^{h'}y^{m'}R_{himj} = -(2g)R_{0i0j}$$

where R_{himj} is Cartan's tensor given by the formula

$$R_{himj} = \partial\Gamma^*_{him}/\partial y^j - (\partial\Gamma^*_{him}/\partial y^{r'})(\partial G^r/\partial y^{j'}) - \partial\Gamma^*_{hij}/\partial y^m$$
$$+ (\partial\Gamma^*_{hij}/\partial y^{r'})(\partial G^r/\partial y^{m'}) + \Gamma^*_{hkj}\Gamma^{*\,k}_{i\ m} - \Gamma^*_{ikj}\Gamma^{*\,k}_{h\ m}$$
$$+ C_{hir}[\partial^2 G^r/\partial y^{m'}\partial y^j - \partial^2 G^r/\partial y^{j'}\partial y^m - (\partial^2 G^r/\partial y^m\partial y^k)(\partial G^k/\partial y^{j'})$$
$$+ (\partial^2 G^r/\partial y^{j'}\partial y^{k'})(\partial G^k/\partial y^{m'})]$$

and R_{oioj} is $\ell^h\ell^m R_{himj}$.

7. The Jacobi differential equations. In considering the Jacobi differential equations we suppose, as does Wren [8, p. 175], that the parameter t is so specialized that $(2g)^{1/2} = 1$. The second variation of equation (6:8) then takes the form

$$(7:1)\quad I''(0) = \int_{t_1}^{t_2} [g_{ij}(D\eta^i/dt)(D\eta^j/dt) - \eta^i\eta^j R_{ij} - (g_i\eta^i)'^2]dt,$$

since from equation (2:5) ℓ_i is now the same as g_i. Introducing the notation

$$2\Omega = g_{ij}(D\eta^i/dt)(D\eta^j/dt) - \eta^i\eta^j R_{ij},$$

we see from equation (6:6) that

$$(7:2)\qquad 2\omega = 2\Omega + d(\eta^i\eta^j y^{m'}\Gamma^*_{imj})/dt.$$

Since the Euler differential equations are identically zero when the integrand function is a derivative, it follows that the Euler differential equations for $I''(0)$ in the form (7:1) are exactly the same as the equations

$$J_i(\eta) = d\,\omega_{\eta i'}/dt - \omega_{\eta i} - (g_j\eta^j)''g_i = 0$$

given by Wren [8, p. 176].

With the aid of equation (7:2) and the statement at the end of the last paragraph it is clear that

$$J_i(\eta) = d\Omega_{\eta i'}/dt - \Omega_{\eta i} - g_i(g_j \eta^j)'' = 0. \tag{7:3}$$

It can be easily verified by means of the definition of Ω and the equation (6:3) that

$$\begin{aligned} \Omega_{\eta i'} &= g_{ij}(D\eta^j/dt), \\ \Omega_{\eta i} &= y^{h'} \Gamma^*_{ijh}(D\eta^j/dt) - R_{ij}\eta^j. \end{aligned} \tag{7:4}$$

Obviously $\Omega_{\eta i'}$ is a covariant tensor of rank one and we may take its covariant derivative. Using (3:5) and the fact that along an extremal with the parameter so chosen that $(2g)^{1/2} = 1$

$$D\ell^i/dt = Dy^{i'}/dt = 0,$$

we may write

$$D\Omega_{\eta i'}/dt = d\Omega_{\eta i'}/dt - \Omega_{\eta k'}\Gamma^*_{i\ h}^{\ k} y^{h'}.$$

By means of postulate (D) and equations (3:15) and (7:4) it follows from the last equation that

$$d\Omega_{\eta i'}/dt = g_{ij}D^2\eta^j/dt^2 + \Gamma^*_{ijh}y^{h'}D\eta^j/dt. \tag{7:5}$$

Substituting from (7:5) and (7:4) in (7:3) we find the following result:

THEOREM 7:1. The first members of the Jacobi equations of the problem of the calculus of variations in the form proposed by Wren are the elements of the tensor

$$J_i(\eta) = g_{ij}D^2\eta^j/dt^2 + R_{ij}\eta^j - g_i(g_j\eta^j)''. \tag{7:6}$$

These equations satisfy along an extremal arc the relation

$y^{i'}J_1(\eta) = 0$ and are not independent, as Wren has shown [8,p. 179]. Wren has also shown [8, pp. 176-7] that all the solutions of the equations $J_1(\eta) = 0$ satisfying the orthogonality condition $g_1\eta^1=0$ are solutions of the equations of Wren [8, p. 177]

$$K_1(\eta) = d\omega_{\eta 1}/dt - \omega_{\eta 1} = 0$$

with initial conditions $g_1\eta^1 = (g_1\eta^1)' = 0$ at t_1. The functions $K_1(\eta)$ are the elements of the tensor

$$K_1(\eta) = g_{1j}D^2\eta^j/dt^2 + R_{1j}\eta^j.$$

8. The determinant $\Delta(t,t_1)$. The determinant $\Delta(t,t_1)$ whose zeros determine the points 3 conjugate to the initial point 1 on the extremal are E_{12} has in Wren's notation [8, p. 184] the form

$$(t,t_1) = \begin{vmatrix} u^{ik}(t) & v^{ik}(t) & y^{i'}(t) & 0 \\ u^{ik}(t_1) & v^{ik}(t_1) & 0 & y^{i'}(t_1) \end{vmatrix}$$

in which the matrices (u^{ik}), (v^{ik}) consist of n - 1 columns of solutions of the equations $K_1(\eta) = 0$. As Wren notes [8, p. 181] the columns may be made into orthogonal solutions [8, p. 174] of $J_1(\eta) = 0$ by subtracting from each suitable multiples of the last two columns. These orthogonal solutions of the Jacobi equations are functions which will be denoted by

$$\eta^i_r(t) \qquad (i = 1,\ldots,n;\ r = 1,\ldots,2n - 2).$$

Since, as equation (7:6) shows, the left-hand members of the Jacobi equations are the components of a covariant tensor, it follows that if η^i_r, r fixed, is a solution of $J_1(\eta) = 0$, its transform $\bar{\eta}^k_r = \eta^i_r \partial\bar{y}^k/\partial y^i$ is a solution of the transformed

equation $\overline{J}_k(\eta) = 0$. We rewrite $\Delta(t,t_1)$ in the form

$$(8:1) \qquad \begin{vmatrix} \eta_r^i(t) & y^{i'}(t) & 0 \\ \eta_r^i(t_1) & 0 & y^{i'}(t_1) \end{vmatrix}$$

If a transformation of variables (2:1) is made with $\bar{\eta}_r^k = \eta_r^i \partial \bar{y}^k / \partial y^i$ and $\bar{y}^{k'} = y^{i'} \partial \bar{y}^k / \partial y^i$, the transform $\bar{\Delta}(t,t_1)$ of the determinant (8:1) becomes

$$\begin{vmatrix} \eta_r^i \partial \bar{y}^k / \partial y^i & y^{i'} \partial \bar{y}^k / \partial y^i & 0 \\ (\eta_r^i \partial \bar{y}^k / \partial y^i)_1 & 0 & (\bar{y}^{i'} \partial \bar{y}^k / \partial y^i)_1 \end{vmatrix}$$

where the parentheses with a subscript 1 indicate that the quantity inside is to be evaluated at $t = t_1$. It can easily be seen that $\bar{\Delta}(t,t_1)$ is the product of the determinant

$$(8:2) \qquad \begin{vmatrix} \partial \bar{y}^k / \partial y^i & 0 \\ 0 & (\partial \bar{y}^k / \partial y^i)_1 \end{vmatrix}$$

by the determinant (8:1). When the determinant (8:2) is evaluated by the Laplace expansion we have the following result:

THEOREM 7:1. *The determinant $\Delta(t,t_1)$ whose zeros determine the points 3 conjugate to the initial point 1 on the extremal arc E_{12} transforms by the law*

$$\bar{\Delta}(t,t_1) = J(\bar{y},y)[J(\bar{y},y)]_1 \Delta(t,t_1)$$

where $J(\bar{y},y)$ is the Jacobian of the transformation $\bar{y} = \bar{y}(y)$. Consequently the transforms of the conjugate points for the problem in terms of the original variables y^i are conjugate points for the problem in the transformed variables $\bar{y}^i$.

9. The canonical differential equations. For the discussion of the Hamilton-Jacobi theory Wren [8, p. 186] replaces the original variables y^i, $y^{i'}$ by canonical variables y^i, z_i in which the z_i are defined for the region R by the equations

$$z_i = g_i(y,y'). \tag{8:1}$$

Since $|g_{ij}| \neq 0$ these equations are solvable, at least locally, for the $y^{i'}$ as functions of y^i and z_i so that

$$y^{i'} = P^i(y,z). \tag{8:2}$$

The Hamilton function may be defined to be

$$H(y,z) = [2g(y,P)]^{1/2}. \tag{8:3}$$

This differs unessentially from Wren's definition. His function is this one minus unity. According to his formula [8, p. 187]

$$H_{y^i} = -g_{y^i}(y,P)/H, \qquad H_{z_i} = P^i(y,z)/H. \tag{8:4}$$

The canonical differential equations are then [8, p. 188]

$$y^{i'} - H_{z_i}(y,z) = 0, \quad z_i' + H_{y^i}(y,z) = 0, \quad H(y,z) = 1. \tag{8:5}$$

It is evident from equations (8:1) and (8:2) that z_i and P^i are the components, respectively, of a covariant and a contravariant tensor of rank one. Hence from (8:4) the H_{z_i} are the components of a contravariant unit tensor. As a result the left-hand members of the first set of equations (8:5) are the components of a contravariant tensor of rank one. A similar statement cannot be made immediately for the second set, since neither the $z_i'(t)$ nor the H_{y^i} are themselves the components of a tensor. However, their sum is a covariant tensor, as will now be shown.

In order to see this we form the absolute derivative Dz_i/dt, the fundamental function being $g = g(y,P)$. We have from equation (3:5)

$$(8:6)\qquad Dz_i/dt = dz_i/dt - z_k\Gamma^{*}{}_{i\ h}^{\ k}\,dy^h/dt - z_k C_i{}^k{}_h\,DP^h/dt.$$

But since $z_k = g_k(y,P)$ it follows from (4:4) that $z_k C_i{}^k{}_h = 0$. Moreover, with P^i in place of $y^{i'}$ in the function g, we see from (1:4) that $z_k = P^i g_{kj}$, and hence equation (8:6) becomes

$$(8:7)\qquad Dz_i/dt = dz_i/dt - P^j\Gamma^{*}{}_{ijh}P^h.$$

When equations (3:7) are multiplied by P^j and P^h and summed it follows from the homogeneity properties (1:4) that

$$(8:8)\qquad Dz_i/dt = dz_i/dt - g_{yi}(y,P).$$

Hence we have the result:

THEOREM 8:1. The left-hand members of the $2n+1$ canonical differential equations (8:5) of Wren may be expressed in the form

$$y^{i'} - H_{z_i}(y,z), \qquad Dz_i/dt, \qquad H(y,z) \equiv [2g(y,P)]^{1/2}.$$

The first two sets of n functions are the components, respectively, of a contravariant and a covariant tensor. The last function, $H(y,z)$, is an invariant.

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VITA

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